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Mode spectrum of the electromagnetic field in open universe models

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ABSTRACT

We examine the mode functions of the electromagnetic field on spherically symmetric backgrounds with special attention to the subclass which allows for a foliation as open Friedmann–Lemaître (FL) space–time. It is well known that in certain scalar field theories on open FL background there can exist so-called supercurvature modes, their existence depending on parameters of the theory. Looking at specific open universe models, such as open inflation and the Milne universe, we find that no supercurvature modes are present in the spectrum of the electromagnetic field. This excludes the possibility for superadiabatic evolution of cosmological magnetic fields within these models without relying on new physics or breaking the conformal invariance of electromagnetism.

Key words: magnetic fields – cosmology: theory – early Universe.

1 INTRODUCTION

The generation of large-scale coherent magnetic fields in the Universe which are observed in low- and high-redshift galaxies (Kronberg 1994; Pentericci et al. 2000), clusters (Clarke, Kronberg & Böhringer 2001), filaments (Battaglia et al. 2009) and even in voids (Neronov & Vovk 2010; Taylor, Vovk & Neronov 2011) is still an unsolved problem in cosmology. Fields generated in the early Universe are generally small and, on large scales, they usually simply evolve via flux conservation, $B \propto a^{-2}$, where a denotes the scale factor of cosmic expansion. One exception to this rule is helical magnetic fields which develop an inverse cascade moving power from small to larger scales (Banerjee & Jedamzik 2004; Campanelli 2007). This can alleviate the problem of magnetic field generation somewhat but is still not sufficient (Caprini, Durrer & Fenu 2009; Durrer, Hollenstein & Jain 2011).

Another idea has been put forward recently in Barrow & Tsagas (2011): in an open universe, supercurvature modes decay slower than $1/a^2$ and can therefore remain relevant at late times. The question remains of how such supercurvature modes are generated. In this paper, we explore this proposal within explicit open universe models. Whilst open inflation is no longer the most favoured model of inflation, it is the most explicit model that leads to an open universe, and we therefore start by studying the generation of supercurvature modes within that model. We show that within the Coleman–de Luccia bubble universe (Coleman & De Luccia 1980; Bucher, Goldhaber & Turok 1995), supercurvature modes of the magnetic field are actually not part of the physical spectrum and can therefore not be generated. We show that the same results also hold for the Milne universe. These are two explicit cases without big bang singularity. For them we can unambiguously specify the initial Cauchy surface needed to define the quantum vacuum and the physical spectrum. Of course we could arbitrarily pronounce any open Friedmann slicing $\{t = \text{const}\}$ as our initial Cauchy surface. But besides the fact that this surface does not allow supercurvature modes, such a definition is arbitrary, incomplete and not unique.

Let us first, in a brief paragraph, present the issue of supercurvature modes. The Laplacian on the spatial slices $\{t = \text{const}\}$ in an open Friedmann universe with curvature K has eigenfunctions with eigenvalues $-k^2$,

$$\Delta Y_k = -k^2 Y_k \,.$$

The functions with $k^2 > |K|$ or, for symmetric, traceless tensors of rank m, with $k^2 > (1 + m)|K|$, form a complete set of functions on these slices which reduces to the usual Fourier modes in the limit $K \to 0$. There are, however, also so-called supercurvature modes, eigenfunctions of the Laplacian with eigenvalues in the range $0 < k^2 < |K|$. At first glance one might argue that, since every square integrable function can be expanded in terms of the subcurvature basis, supercurvature modes play no role. If we consider only the post-inflationary Universe, this view seems justified. However, it has been shown in Sasaki, Tanaka & Yamamoto (1995) that in open inflation under certain circumstances supercurvature modes can be present, see also Lyth & Woszczyna (1995) and García-Bellido et al. (1995) for a discussion in a different context. The basic reason for this is that the $\{t = \text{const}\}$ slices of the post-inflationary universe do not represent Cauchy surfaces of the entire space–time containing the Coleman–de Luccia bubble. However, in order to

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discuss quantum fields and particle generation during inflation, we have to expand the fields in a complete basis on a Cauchy surface of the inflationary universe and, as has been shown in Sasaki et al. (1995), in certain cases this can lead to the generation of modes which correspond to supercurvature modes after inflation.

In Sasaki et al. (1995), the analysis is presented for scalar fields. In this paper, we reduce the case of the electromagnetic field to the scalar field problem so that we can apply the results of Sasaki et al. (1995). We show that when expressing the electromagnetic field in terms of the Debye potentials (Mo & Papas 1972), these can be viewed as two conformally coupled fields for which no supercurvature modes exist. A definite statement can be made in any setup which allows for the Cauchy problem to be well posed. On physical grounds, although there may be no unique mechanism to procure an open Friedmann–Lemaître (FL) universe, we want to focus on the open inflation scenario. We think that a case study within this scenario is most useful because it is by far the most explicit and physically well-motivated setup which naturally leads to an open FL universe and, at the same time, comes with a complete description in the Cauchy sense. Other scenarios which, e.g., impose an open geometry to be realized ad hoc have to be supplemented by some arbitrary assumptions, and the question of supercurvature modes can therefore not seriously be addressed.

The open inflation scenario (Bucher, Goldhaber & Turok 1995; García-Bellido, Garriga & Montes 1998) was originally introduced at a time when observational data seemed to favour an open Universe and it was therefore imperative to look for appropriate models. With the advent of precision measurements of the anisotropies in the cosmic microwave background, the evidence for considerable curvature to be present in our Universe has virtually evaporated (Jaffe et al. 2001; Spergel et al. 2003). However, the scenario has recently attracted new interest in the context of eternal inflation (Linde 1986; Guth 2007) and the landscape idea (Susskind 2007; Weinberg 2007). From this new point of view, open inflation in the Coleman–de Luccia bubble Universe remains conceptually well motivated, although the focus has shifted away from procuring non-vanishing curvature. In fact, the scenario allows that the curvature we observe today can be rather minuscule, see e.g. De Simone & Salem (2010) for a discussion. As already pointed out, in this paper we exploit the fact that the setting contains enough information about the background space–time such that the questions we want to study can be addressed in a meaningful way. Here, we are not so much interested in the question of whether the spatial curvature of the observed Universe is negative, but we want to analyse the conceptual question of whether an open universe can allow for supercurvature modes of the electromagnetic field.

The remainder of this paper is organized as follows: in the next section we introduce the Debye potentials, write the electromagnetic Lagrangian in an open, closed or flat FL universe in these variables and derive a complete set of solutions to the Euler–Lagrange equations. We then analyse whether supercurvature modes are normalizable on a Cauchy surface, and by comparison with the pure scalar case, we conclude that no supercurvature modes are normalizable on an open de Sitter geometry. By conformal invariance we find that the same result holds for a Coleman–de Luccia bubble. Finally, for completeness, we provide the full quantization prescription of the Debye potentials, before summarizing our results. In Appendix A, we present the explicit computation of the mode normalization also for the Milne model, which is one of the simplest open universe models.

2 GENERAL FORMALISM

We consider background geometries of the FL type. The line element reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}r^2 + s^2(r)\mathrm{d}\Omega^2], \tag{1}$$

where $s(r) = \sin r$, r and $\sinh r$ corresponds to closed, flat and open spatial geometry, respectively. In this work, we will focus on the latter two cases. In particular, the case $a(t) \equiv \text{const}$, $s(r) \equiv r$ gives the Minkowski metric, while $a(t) \equiv \sinh(Ht)/H$, $s(r) \equiv \sinh r$ represents an open foliation of de Sitter space with $\Lambda = 3H^2$. Note that with this convention, r and s have no units and spatial curvature is $K = \pm 1$ or 0, but a and t have units of length. As usual, we set $c = \hbar = 1$.

Because these types of backgrounds are spherically symmetric, they are appropriate for studying the electromagnetic field in terms of the Debye potentials (Mo & Papas 1972). In this formalism, instead of making use of the usual $\boldsymbol{A}^\mu$ vector potential, the electromagnetic field is decomposed into two potentials U and V. The advantage of these Debye potentials is the fact that the equations completely decouple in any spherically symmetric background, while the components of $\boldsymbol{A}^\mu$ are usually badly mixed if the space–time is not flat. Therefore, the Debye potentials allow for even more general metrics than (1).

In equations (4) and (5) of Mo & Papas (1972), expressions for the physical electric and magnetic fields are given in terms of the Debye potentials. It will be useful for us to look at these fields in the helicity basis. Given an orthonormal basis $(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ on the sphere, the helicity basis reads

$$\boldsymbol{e}_+ = \frac{1}{\sqrt{2}}(\boldsymbol{e}_\theta - \mathrm{i}\boldsymbol{e}_\phi), \qquad \boldsymbol{e}_- = \frac{1}{\sqrt{2}}(\boldsymbol{e}_\theta + \mathrm{i}\boldsymbol{e}_\phi). \tag{2}$$

We find the following components of the physical electric and magnetic field in this new basis:

$$\begin{aligned}
E_r &= -\frac{1}{2as}(\eth\eth^* + \eth^*\eth)V, \\
B_r &= -\frac{1}{2as}(\eth\eth^* + \eth^*\eth)U, \\
E_+ &= -\frac{1}{\sqrt{2}as}[\partial_r(s\eth V) + \mathrm{i}\partial_t(as\eth U)], \\
B_+ &= -\frac{1}{\sqrt{2}as}[\partial_r(s\eth U) - \mathrm{i}\partial_t(as\eth V)], \\
E_- &= E_+^*, \qquad B_- = B_+^*.
\end{aligned} \tag{3}$$

In these expressions, we make use of the *spin-raising* and *spin-lowering* operators $\eth$ and $\eth^*$. These are defined as

$$\begin{aligned}
\eth\chi &= -\sin^\sigma\theta\,\partial_\theta(\sin^{-\sigma}\theta\,\chi) - \frac{\mathrm{i}}{\sin\theta}\partial_\phi\chi, \\
\eth^*\chi &= -\sin^{-\sigma}\theta\,\partial_\theta(\sin^\sigma\theta\,\chi) + \frac{\mathrm{i}}{\sin\theta}\partial_\phi\chi,
\end{aligned} \tag{4}$$

where σ is the spin weight of the field χ. As the names imply, the spin-raising and spin-lowering operators increase or decrease the spin weight of a field by one unit. See, e.g., Goldberg et al. (1967) for some details concerning these operators and the spherical harmonics used to expand a field of arbitrary spin weight.

Using $F_{\mu\nu}F^{\mu\nu} = 2\boldsymbol{B}^2 - 2\boldsymbol{E}^2$, the Maxwell action in terms of the Debye potentials is

$$\mathcal{S}_{\rm em} = \frac{1}{2}\int a^3 \mathrm{d}t s^2 \mathrm{d}r \mathrm{d}\Omega \left[-\frac{1}{a^2}\partial_t(a\eth U)\partial_t(a\eth^* U) + \frac{1}{a^2 s^2}(\eth^*\eth U)(\eth\eth^* U) + \frac{1}{a^2 s^2}\partial_r(s\eth U)\partial_r(s\eth^* U)\right] - \{U \to V\}. \tag{5}$$

It is evident from the action that the physical degrees of freedom are carried by the helicity-one representations $\eth U$ and $\eth V$. This is in agreement with the well-known properties of a photon. Furthermore, it is not surprising that the two helicity degrees of freedom decouple in a spherically symmetric background.

The equations of motion are[1]

$$\frac{1}{a^3}\partial_t(a^3\partial_t U) - \frac{1}{a^2}\Delta U + \left(\frac{(\partial_t a)^2}{a^2} + \frac{\partial_t^2 a}{a} - \frac{\partial_r^2 s}{a^2 s}\right) U = \Box U + m_{\rm eff}^2 U = 0, \tag{6}$$

and the same for V. Since U and V have identical properties, from now on we will only focus on U. All results apply to V in exactly the same way. In the above expression, we have introduced the spatial Laplace operator Δ for a scalar field defined as

$$\Delta = \frac{1}{s^2}\partial_r(s^2\partial_r) + \frac{1}{2s^2}(\eth\eth^* + \eth^*\eth). \tag{7}$$

The operator $\Box$ in (6) is precisely the d'Alembertian for a Lorentz scalar. However, since U is not itself a Lorentz scalar, it also acquires an effective mass $m_{\rm eff}$, which is precisely the one that corresponds to a conformal coupling to curvature.

In the case of spatial flatness, a complete set of eigenfunctions to Δ is given by

$$X_{p\ell m}(r,\theta,\phi) = p\sqrt{2/\pi}\, j_\ell(pr) Y_{\ell m}(\theta,\phi), \tag{8}$$

where we have chosen the spherical Bessel functions j_ℓ which are regular at the origin. The eigenvalue equation on the flat three-dimensional surface is

$$-\Delta X_{p\ell m} = p^2 X_{p\ell m}, \tag{9}$$

and the eigenfunctions are normalized as

$$\int \mathrm{d}r r^2 \mathrm{d}\Omega X_{p\ell m} X^*_{p'\ell' m'} = \delta(p-p')\delta_{\ell\ell'}\delta_{mm'}. \tag{10}$$

In the case of an open geometry where $s = \sinh r$, the eigenfunctions are the harmonics on the three-hyperboloid. The eigenvalue equation reads

$$-\Delta Y_{p\ell m} = (p^2+1) Y_{p\ell m}, \tag{11}$$

where the eigenfunctions $Y_{p\ell m}$ which are regular at $r = 0$ are given by

$$Y_{p\ell m}(r,\theta,\phi) = f_{p\ell}(r) Y_{\ell m}(\theta,\phi),$$
$$f_{p\ell}(r) \equiv \frac{\Gamma(\mathrm{i}p+\ell+1)}{\Gamma(\mathrm{i}p+1)}\frac{p}{\sqrt{\sinh r}} P^{-\ell-1/2}_{\mathrm{i}p-1/2}(\cosh r), \tag{12}$$

see, e.g., Sasaki et al. (1995). The normalization is again such that

$$\int \mathrm{d}r \sinh^2 r \mathrm{d}\Omega Y_{p\ell m} Y^*_{p'\ell' m'} = \delta(p-p')\delta_{\ell\ell'}\delta_{mm'}, \tag{13}$$

this time on the three-hyperboloid.

The Debye potentials are conformally coupled to gravity, meaning that any conformal factor *which preserves the spherical symmetry of the geometry* can be absorbed into a redefinition of the fields. In particular this means that the equation of motion (6) is invariant under a time-dependent conformal rescaling $g_{\mu\nu} \to \omega^2(t) g_{\mu\nu}$ (with the corresponding redefinition of time) and a simple rescaling of the field as $U \to \omega^{-1}(t) U$.

3 NO SUPERCURVATURE MODES

A special situation is given in the open universe models because the three-hyperboloids used in the foliation do not usually represent global Cauchy surfaces. Therefore, the failure of modes to be normalizable on the hyperboloids does not necessarily imply that they should be excluded from the physical spectrum. What really matters is the question of whether or not a mode is normalizable on a Cauchy surface. It is well known that in certain scalar field models this leads to the occurrence of modes with discrete imaginary values of p in the spectrum which are usually referred to as *supercurvature modes*.

In the case of the magnetic field, it was found in Barrow & Tsagas (2011) – see also references therein to earlier work, e.g., Barrow & Tsagas (2008) and Tsagas & Kandus (2005) – that supercurvature modes, if they exist, give rise to superadiabatic evolution and can therefore help to solve the problem of magnetogenesis. It is therefore of relevance whether or not the electromagnetic field can support supercurvature modes. With the formalism of the Debye potentials, we can now easily address this question.

In order to study whether such supercurvature modes are relevant, we have to check if there are some modes with imaginary p which are normalizable on a Cauchy surface. As mentioned before, such a surface can usually not be found within the patch covered by the open coordinate chart. One therefore has to complete this chart, which means that one usually has to continue the coordinates across the initial singularity of the open chart (which is a coordinate singularity). As a specific example which is general enough, one can consider the creation of an open universe by the Coleman–de Luccia process (Coleman & De Luccia 1980), as in the open inflation scenario (Bucher et al. 1995; García-Bellido et al. 1998). A space–time diagram is shown in Fig. 1. In this type of model, the open region (indicated as region I in Fig. 1) is contained fully within the lightcone of the nucleation event of a bubble which was created by a vacuum metastability transition. However, the entire one-bubble space–time can easily be constructed from the instanton which is responsible for the transition. A Cauchy surface is then given, e.g., by the maximal three-section of the instanton, which represents the so-called turning-point geometry (Coleman & De Luccia 1980). It is located along the horizontal line indicated as Σ_0 in the figure. Any time evolution of this surface is, of course, equally suitable.

[1] As in equations (6) and equation (7) of Mo & Papas (1972), we have omitted an overall spherical Laplacian $(\eth\eth^* + \eth^*\eth)/2$ acting on the equation. The solutions are identical up to modes which are annihilated by this operator. These are exactly the modes of U with zero angular momentum ($\ell = 0$). Noting that these modes are already annihilated by $\eth$ and $\eth^*$ individually, it is evident from inspecting equation (3) that they are pure gauge modes which do not contribute to the physical electromagnetic field. Note also that with this operator, the equations of motion would appear to be fourth order in the angular coordinates. The equations for the true physical degrees of freedom $\eth U$ and $\eth V$, however, would remain second order in all coordinates.

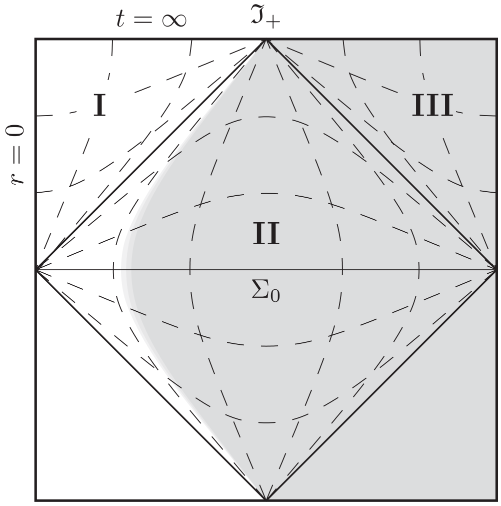

Figure 1. Space–time diagram of a one-bubble geometry which may be the result of a Coleman–de Luccia process. If one can neglect the geometric effect of the bubble (indicated as the white region), the space–time is approximately de Sitter. There always exists a conformal map, given as a finite conformal factor, between the $O(3, 1)$-symmetric one-bubble space–time and the $O(4, 1)$-symmetric de Sitter space. Region I resembles an open FL universe but does not contain any global Cauchy surface. Such a surface is indicated as Σ_0, which is entirely contained in region II. Region III is another open FL universe, similar to, but causally disconnected from region I. Some surfaces of constant radial or constant time coordinate are indicated as dashed lines.

In order to make the problem tractable analytically, let us ignore for a moment the geometric effects of the bubble altogether. That is, we consider an exact de Sitter geometry with $a(t) \equiv \sinh(Ht)/H$, $s(r) \equiv \sinh r$. For de Sitter space, the question of supercurvature modes (in a scalar field setting) has been thoroughly studied in Sasaki et al. (1995). It turns out that their analysis can be easily applied to the present setup. Comparing our equation (6) with the equation (2.7) of Sasaki et al. (1995), it is evident that the ν-parameter is the one of the conformally coupled field. This should, of course, not come as a surprise since the electromagnetic field is conformally coupled. The analysis then proceeds with the calculation of the normalization of the modes on a Cauchy surface. While Sasaki et al. (1995) work in de Sitter space, we will present this calculation for an even simpler toy model of an open FL universe, the Milne model, as a pedagogical example in Appendix A.

It is shown in Sasaki et al. (1995) that *no supercurvature modes exist* in the conformally coupled case. By re-applying exactly the same arguments we can therefore conclude that there are no supercurvature modes for the Debye potentials as well. The result, so far, holds for the case of an exact open de Sitter background. However, one can show that it rigorously holds also for any one-bubble geometry like the ones produced in an arbitrary Coleman–de Luccia process. This can be seen by noting two facts. First, the radial coordinate on the Cauchy surface corresponds to the analytic continuation of the time coordinate of the open chart. The continuation of the scale factor a into the Euclidean domain therefore characterizes the geometric effects of the bubble. It is the behaviour in Euclidean time which determines the normalizability of a mode. Secondly, we note that any bubble geometry can be mapped on to an exact de Sitter geometry by a finite conformal factor, which may depend on the radial coordinate on the Cauchy surface. However, we have already pointed out that the mode equation is invariant under such a conformal transformation. In particular, the normalizability of a mode is not affected by any finite conformal rescaling. In fact this means that any $O(3, 1)$-symmetric geometry has the same spectrum of modes for the electromagnetic field. The non-existence of electromagnetic supercurvature modes then follows as a corollary from Sasaki et al. (1995).

Since the formalism we apply here is very different, it is worthwhile to explain the connection to Barrow & Tsagas (2011) in some more detail. Given our expression (3) for the magnetic field and using the mode expansion of equation (12) for U and V, one can show that the covariant three-dimensional (spatial) vector Laplacian acting on a magnetic mode with wavenumber p yields

$$-\Delta \boldsymbol{B}_{(p)} = \frac{p^2+2}{a^2}\boldsymbol{B}_{(p)}. \tag{14}$$

A comparison with equation (7) of Barrow & Tsagas (2011) (see also their footnote 6) then clarifies the relation between our wavenumber p and their eigenvalue parametrization n. The superadiabatic modes with eigenvalues $n^2 < 2$ correspond to imaginary wavenumbers p. We have just shown that these modes are not included in the spectrum in any one-bubble open universe scenario.

4 QUANTUM THEORY

A canonical quantization prescription for the Debye potentials works as follows. First, we note that the true physical degrees of freedom which should be quantized are given by $\eth U$ and $\eth V$. Then, it is advised to rescale the fields such that the Hubble damping term in the mode equation disappears. To this end, we write $\eth U \equiv \upsilon/a$ and choose a conformal time coordinate defined by $\mathrm{d}t = a\mathrm{d}\tau$. The action for the rescaled field υ reads

$$\mathcal{S}_\upsilon = \frac{1}{2}\int \mathrm{d}\tau s^2 \mathrm{d}r \mathrm{d}\Omega \bigg[-\partial_\tau \upsilon \partial_\tau \upsilon^* + \frac{1}{s^2}\eth^* \upsilon \eth \upsilon^* + \frac{1}{s^2}\partial_r(s\upsilon)\partial_r\left(s\upsilon^*\right)\bigg]. \tag{15}$$

Following the usual rules of canonical quantization, the field υ is promoted to an operator $\hat{\upsilon}$ and can be expanded in terms of creation and annihilation operators of modes by writing

$$\hat{\upsilon}(\tau, r, \theta, \phi) = \int \mathrm{d}p \sum_{\ell m} \frac{1}{\sqrt{2}}\Big[\hat{\mathrm{a}}_{p\ell m}\upsilon_p(\tau)\,{}_1Y_{p\ell m}(r, \theta, \phi) + \hat{\mathrm{a}}^\dagger_{p\ell m}\upsilon^*_p(\tau)\,{}_1Y^*_{p\ell m}(r, \theta, \phi)\Big]. \tag{16}$$

Note that the field υ is of spin weight 1 and should therefore be expanded in terms of the appropriate spherical harmonics. The eigenfunctions ${}_1Y_{p\ell m}$ of the spatial Laplace operator are the ones of equation (12), with $Y_{\ell m}$ replaced by the corresponding spherical harmonic of spin-weight 1, ${}_1Y_{\ell m}$. In the case of spatial flatness, the eigenfunctions ${}_1Y_{p\ell m}$ have to be replaced by ${}_1X_{p\ell m}$, which are related to equation (8) in a similar way. In both cases, the mode functions $\upsilon_p(\tau)$ are governed by the mode equation

$$\partial_\tau^2 \upsilon_p + p^2 \upsilon_p = 0, \tag{17}$$

and it is allowed to choose them as independent of ℓ and m. If one uses normalized mode functions

$$\mathrm{Im}(\upsilon_p \partial_\tau \upsilon_p^*) = 1, \tag{18}$$

then the equal time commutator of field and canonical momentum,

$$[\hat{\upsilon}(\tau, r, \theta, \phi), \partial_\tau \hat{\upsilon}^*(\tau, r', \theta', \phi')] = \mathrm{i}\frac{1}{s^2}\delta(r-r')\delta(\cos\theta - \cos\theta')\delta(\phi-\phi'), \tag{19}$$

is equivalent to the standard commutation rules for the creation and annihilation operators,

$$\left[\hat{a}_{p\ell m}, \hat{a}^{\dagger}_{p'\ell' m'}\right] = \delta(p - p')\delta_{\ell\ell'}\delta_{mm'}. \tag{20}$$

The reader may wonder what would have been the difference if one had quantized the scalar Debye potentials directly instead of the helicity-one degrees of freedom which we obtained by applying a spin-raising operator. First, by choosing to quantize the latter, we have avoided a quantization of the unphysical gauge modes with $\ell = 0$ which are present in the expansion of a scalar but not in the one of the helicity-one fields. Secondly, by noting that $\eth Y_{\ell m} = \sqrt{\ell(\ell+1)}\,{}_1Y_{\ell m}$, one can see that some ℓ-dependent factors may appear in the commutation rules for the scalar Debye potentials. A careful look at the action reveals that the scalar modes are not canonically normalized and that these factors are therefore expected. These differences, however, are completely inessential for the question of whether or not any supercurvature modes are part of the spectrum. In particular, the argument of Section 3 works equally well for the helicity-one degrees of freedom, with rather obvious modifications when going through the detailed proof.

The standard choice of positive frequency mode functions is like in Minkowski space,

$$\upsilon_p(\tau) = \frac{1}{\sqrt{p}} \mathrm{e}^{-\mathrm{i}p\tau} \qquad \text{(Minkowski)}. \tag{21}$$

This is not surprising as the rescaled electromagnetic fields, B/a^2 and E/a^2, are independent of the scale factor in conformal time. One can verify that some standard results of quantized electromagnetism are reproduced. We checked this for the vacuum two-point correlators $\langle E_a(t, r, \theta, \phi)E_b(t', r', \theta', \phi')\rangle$, which turn out to be the same as if obtained from a standard quantization of A^μ in Minkowski space.

However, the above quantization prescription is general enough to be applicable also in arbitrary flat or open FL backgrounds. For instance, one could obtain the primordial power spectrum of the electromagnetic field in the open inflation scenario. Since we have proven that no supercurvature modes are present, the result of this computation is, however, of pure academic interest, because one does not expect that significant perturbation amplitudes can be obtained after the inflationary era.

5 CONCLUSIONS

In this paper, we have studied the modes of the quantum vacuum of the electromagnetic field in an open, inflating Friedmann universe. Whilst subcurvature modes decay extremely fast after their generation, supercurvature modes, on the other hand, could remain relevant at the end of inflation and could then have a significant impact on the origin of large-scale magnetic fields in the present Universe. Great care should therefore be taken in understanding under which circumstances supercurvature modes are expected to belong to the magnetic field spectrum. Here we have explored the eigenmodes of the electromagnetic field in an open universe and have shown that supercurvature modes are not expected to be produced via any causal process, such as if the open universe is generated by bubble nucleation. This is a consequence of the conformal coupling of electromagnetism.

If one switches on some perturbative interaction later during inflation, conformal invariance may be broken and electromagnetic modes may be generated. But only modes which are present in the quantum vacuum can be excited by such a perturbative coupling, namely subcurvature modes.

We have focused on the open inflation scenario with the remark that it presently is the only physically motivated scenario which procures an open Friedmann universe and offers a complete enough framework to address the question of supercurvature modes. It also represents the scenario preferred by recent considerations of eternal inflation and the landscape of string theory. Our argument shows already that no supercurvature modes can exist in all cases where the global space–time carries the same $O(3, 1)$-symmetry as is displayed by the open patch. In less specific settings which are deprived of a global description, the question may be elusive, but the experience with the open inflation scenario teaches us that the physical viability of supercurvature modes in the context of standard electromagnetism has yet to be demonstrated. In a singular open Friedmann universe this question cannot be seriously addressed, and when addressed naively, choosing $\{t = \text{const}\}$ hypersurfaces, again no supercurvature modes are contained in the physical spectrum.

In view of this result, it appears more and more unlikely that standard electromagnetism can support any superadiabatic evolution of cosmological magnetic fields under physical conditions. Such an evolution can be obtained only by breaking the conformal invariance of electromagnetism (already at the time of bubble nucleation) or by introducing some other kind of new physics.

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APPENDIX A: MODE SPECTRUM IN THE MILNE UNIVERSE

We present here the explicit computation of the mode spectrum in the Milne model, which is one of the simplest open FL geometries. It is obtained by rewriting the line element of Minkowski space as

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}R^2 + R^2\mathrm{d}\Omega^2 = -\mathrm{d}t^2 + t^2[\mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\Omega^2]. \tag{A1}$$

While $-\infty < T < \infty$ and $0 \leq R < \infty$ are the coordinates of a standard (spatially flat) spherical coordinate system which covers the full Minkowski space–time, using $0 < t < \infty$ and $0 \leq r < \infty$ one obtains a metric of the open FL type with $a = t$, cf. equation (1). This new coordinate system, with $T = t\cosh r$ and $R = t\sinh r$, covers the interior of the future lightcone of $T = R = 0$. Within this very simple setting which yet has all the desired features, we want now to exemplify the reasoning of Section 3.

As a first step, for the mode expansion of equation (12) we can immediately solve the mode equation. The solutions to equation (6) take the form

$$U_{p\ell m\pm}(t, r, \theta, \phi) = N_{p\pm} t^{-1\pm \mathrm{i}p} Y_{p\ell m}(r, \theta, \phi), \tag{A2}$$

where $N_{p\pm}$ is a normalization to be determined. To this end, we want to evaluate the Klein–Gordon inner product on a Cauchy surface. The whole point is that the open spatial hypersurfaces $\{t = \mathrm{const}\}$ do not represent proper Cauchy surfaces and one should therefore make a better choice. We choose the surface $\{T = 0\}$ which is a proper global section of Minkowski space. This hypersurface lies entirely outside the coordinate patch covered by t, r; however, by making appropriate analytic continuations, we can complete the chart to include $\{T = 0\}$. More precisely, by taking $t \rightarrow \mathrm{i}\rho$, $r \rightarrow \tau - \mathrm{i}\pi/2$, the region outside the lightcone is covered by $-\infty < \tau < \infty$ and $0 < \rho < \infty$. Furthermore, the hypersurface $\{T = 0\}$ coincides with the one defined by $\{\tau = 0\}$. The line element is given as

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - \rho^2\mathrm{d}\tau^2 + \rho^2\cosh^2\tau\,\mathrm{d}\Omega^2. \tag{A3}$$

It is noteworthy that the role of time and radial distance have been interchanged by the analytic continuation, just as it was done by Sasaki et al. (1995) for the case of de Sitter. The Klein–Gordon inner product is finally given by

$$\begin{aligned}\langle U_{p\ell m\pm}, U_{p'\ell' m'\pm}\rangle_{\mathrm{K\text{–}G}} &= \mathrm{i}\int_{T=0} \mathrm{d}R\,R^2\mathrm{d}\Omega\, U^*_{p\ell m\pm}\, \overset{\leftrightarrow}{\partial}_T\, U_{p'\ell' m'\pm} \\ &= \mathrm{i}\delta_{\ell\ell'}\delta_{mm'} N^*_{p\pm} N_{p'\pm} \mathrm{e}^{\mp(p+p')\pi/2} \int_0^\infty \frac{\mathrm{d}\rho}{\rho}\rho^{\mp\mathrm{i}(p-p')} \\ &\quad \times \cosh^2\tau\; f^*_{p\ell}\, \overset{\leftrightarrow}{\partial}_\tau\, f_{p'\ell'}\Big|_{\tau=0}.\end{aligned} \tag{A4}$$

By making a change of variables to $\ln\rho$, one can see that the ρ-integral is a representation of the delta function $\delta(p - p')$ for p, p' real. For any imaginary p or p', the integral is badly divergent, which implies that the modes with imaginary p have zero norm. In other words, there are no supercurvature modes of the Debye potentials in the Milne model.

For the regular modes with real values of p, the term printed in the last line of equation (A4) can be easily evaluated once setting $p = p'$ and $\ell = \ell'$. One obtains

$$\cosh^2\tau\; f^*_{p\ell}\, \overset{\leftrightarrow}{\partial}_\tau\, f_{p\ell}\Big|_{\tau=0} = -\frac{2\mathrm{i}p}{\pi}\sinh\pi p. \tag{A5}$$

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